B. Sc. Part III

5th Paper

1. verify stoke's theorem for

$\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c is its boundary

Soln → we've to show that

$$\int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

∵ c is the boundary of upper half surface of sphere $x^2+y^2+z^2=1$.

⇒ c is a circle $x^2+y^2=1$ in the xy-plane.

Let its parametric equations are

$x = \cos t,\ y = \sin t.$

Now $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

Also $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

In xy-plane, $z=0 \Rightarrow dz=0$

$\Rightarrow \vec{F}\cdot d\vec{r} = (2x-y)dx - yz^2 dy = (2x-y)dx$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C (2x-y)\,dx$$

$$= \int_C (2x-y)\frac{dx}{dt}\,dt$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt \quad \left[\because \begin{array}{l} x=\cos t \\ y=\sin t \end{array}\right]$$

$$= -\int_0^{2\pi} 2\sin t\cos t\,dt + \int_0^{2\pi} \sin^2 t\,dt$$

$$= \left[\frac{\cos 2t}{2}\right]_0^{2\pi} + \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt$$

$$= \frac{1}{2}(\cos 4\pi - \cos 0) + \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi$$

Now $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$\Rightarrow \text{curl}\,\vec{F} = \vec{i}(-2yz+2yz) - \vec{j}(0-0) + \vec{k}(0+1) = \vec{k}$

$\Rightarrow \text{curl}\,\vec{F}\cdot\vec{n} = \vec{k}\cdot\vec{n}$

$$\Rightarrow \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \int_S \vec{k}\cdot\vec{n}\,ds = \iint_R \vec{k}\cdot\vec{n}\,\frac{dx\,dy}{\vec{n}\cdot\vec{k}}$$

where R is the projection of S on xy-plane.

$$\Rightarrow \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$$= 4\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}} dy\right] dx$$

$$\Rightarrow \int_S \text{curl}\, \vec{F} \cdot \vec{n}\, ds = 4\int_0^1 \sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\left[\left(0 + \frac{1}{2}\sin^{-1}1\right) - \left(0 + \frac{1}{2}\sin^{-1}0\right)\right]$$

$$= 4 \times \left(\frac{1}{2} \times \frac{\pi}{2} - 0\right) = \pi.$$

$$\Rightarrow \int_S \text{curl}\, \vec{F} \cdot \vec{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

Hence, Stoke's theorem is verified.